HARMONIC FUNCTIONS OF GENERAL GRAPH LAPLACIANS

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Abstract. We study harmonic functions on general weighted graphs which allow for a compatible intrinsic metric. We prove an L^p Liouville type theorem which is a quantitative integral L^p estimate of harmonic functions analogous to Karp's theorem for Riemannian manifolds. As corollaries we obtain Yau's L^p-Liouville type theorem on graphs, identify the domain of the generator of the semigroup on L^p and get a criterion for recurrence. As a side product, we show an analogue of Yau's L^p Caccioppoli inequality. Furthermore, various quantitative results for graphs of finite measure are obtained.

1. Introduction

The study of harmonic functions is a fundamental topic in various areas of mathematics. An important question is which subspaces of harmonic functions consist only of constant functions. Such results are referred to as Liouville type theorems. In Riemannian geometry L^p-Liouville type theorems for harmonic functions were studied for example by Yau [Yau76], Karp [Kar82], Li-Schoen [LS84] and many others. Karp's criterion was later generalized by Sturm [Stu94] to the setting of strongly local regular Dirichlet forms. Over the years there were various attempts to realize an analogous theorem for graphs, see Holopainen-Soardi [HS97], Rigoli-Salvatori-Vignati [RSV97], Masamune [Mas09] and most recently Hua-Jost [HJ13]. In all these works normalized Laplacians were studied (often with further restrictions on the vertex degree) and various criteria, all weaker than Karp's integral estimate, were obtained. The main challenge when considering graphs is the non-existence of a chain rule and, moreover, the fact that for unbounded graph Laplacians the natural graph distance is very often not the proper analogue to the Riemannian distance in manifolds. In this paper, we use the newly developed concept of intrinsic metrics on graphs to prove an analogue to Karp's theorem for general Laplacians on weighted graphs. Thus, we generalize all earlier results on graphs not only with respect to the generality of the setting but also by recovering the precise analogue of Karp's criterion.

In what follows we first state and discuss our results and refer for details and precise definitions to Section 2. Our framework is the one of weighted graphs over a discrete measure space (X, m) introduced in [KL12] which includes non locally finite graphs. In this setting a pseudo metric is called *intrinsic* if the energy measures of distance functions can be estimated by the measure of the graph (see Definition 2.2). We further call such a pseudo metric *compatible* if the weighted vertex degree is bounded on each distance ball and the pseudo metric has finite jump size (see Definition 2.3). As the boundedness of the weighted vertex degree is implied by finiteness of distance balls which is equivalent to metric completeness in the case of a path metric on a locally finite graph, see [HKMW, Theorem A.1], this

assumption can be seen as an analogue of completeness in the Riemannian manifold case. Similarly, Sturm [Stu94] asks for precompactness of balls.

Our main result is the following analogue to Karp's L^p Liouville theorem [Kar82, Theorem 2.2], whose proof is given in Section 3.2. A function is called (sub)harmonic if it is in the domain of the formal Laplacian and the formal Laplacian applied to this function is pointwise (less than or) equal to zero, (see Definition 2.1).

Theorem 1.1 (Karp's L^p Liouville theorem)**.** *Assume a connected weighted graph allows for a compatible intrinsic metric. Then every non-negative subharmonic function f satisfying*

$$\inf_{r_0>0}\int_{r_0}^{\infty}\frac{r}{\|f1_{B_r}\|_p^p}dr=\infty,$$

for some $p\in(1,\infty)$, is constant.

Clearly, the integral in the theorem above diverges, whenever $0\neq f\in L^p(X,m)$. Thus, as an immediate corollary, we get Yau's L^p Liouville type theorem [Yau76].

Corollary 1.2 (Yau's L^p Liouville theorem)**.** *Assume a connected weighted graph allows for a compatible intrinsic metric. Then every non-negative subharmonic function $f\in L^p(X,m)$, $p\in(1,\infty)$, is constant.*

Remark 1.1. (a) The results above imply the corresponding statement for harmonic functions by the simple observation that f_+, f_- and $|f|$ of a harmonic function f are non-negative and subharmonic.

(b) Harmonicity of a function is independent of the choice of the measure m. Hence, for any non-constant harmonic function f on X, we may find a sufficiently small measure m such that $f\in L^p(X,m)$ for any $p\in(0,\infty)$, see [Mas09]. Our theorem states that if we impose the restriction of compatibility on the measure and the metric, then the L^p Liouville theorem holds for $1<p<\infty$.

(c) Theorem 1.1 generalizes all earlier results on graphs [HS97, RSV97, Mas09, HJ13] for the case $p\in(1,\infty)$. Not only that our setting is more general – as the natural graph distance is always a compatible intrinsic metric to the normalized Laplacian – but also our criterion is more general. If f satisfies

$$\limsup_{r\to\infty}\frac{1}{r^2\log r}\|f1_{B_r}\|_p^p<\infty,$$

then the integral in Theorem 1.1 diverges. Thus, Theorem 1.1 implies [HJ13, Theorem 1.1] (which had only r^2 rather than $r^2\log r$ in the denominator). For the normalized Laplacian the case $p\in(1,2]$ can already be obtained by the techniques of [HJ13] (see Remark 3.3 therein). Here, the missing cases $p\in(2,\infty)$ are treated by using a more subtle mean value inequality from [HS97].

(d) Contrary to the normalized Laplacian, [HJ13, Theorem 1.2], there is no L^1 Liouville type theorem in the general case. Counter-examples are given in Section 4 which complement the counter-examples from manifolds, [Chu83, LS84].

(e) In [KL12] discrete measure spaces (X,m) with the assumption that every infinite path has infinite measure are discussed (this assumption is denoted by (A) in [KL12]). It is not hard to see that for connected graphs over (X,m) every non-negative subharmonic function $L^p(X,m)$, $p\in[1,\infty)$ is trivial. From every non-constant positive subharmonic function we can extract a sequence of vertices such that the function values increase along this sequence (compare [KL12, Lemma 3.2

and Theorem 8]). Since this path has infinite measure, the function is not contained in $L^p(X,m)$, $p \in [1,\infty)$. Thus, the only interesting measure spaces are the ones which contain an infinite path of finite measure.

(f) Sturm [Stu94] proves an analogue for Karp's theorem for weakly subharmonic functions. This might seem stronger, however, in our setting on graphs weak solutions of equations are automatically solutions, [HKLW12, Theorem 2.2 and Corollary 2.3].

(g) The techniques of [HJ13] would also allows to recover [HJ13, Theorem 1.1] for the cases $p < 1$.

Corollary 1.2 allows us to explicitly determine the domain of the generator L_p of the semigroup on L^p. We denote by Δ the formal Laplacian with formal domain F (for definitions see Section 2.2). The proof of the corollary below is given in Section 3.4.

Corollary 1.3 (Domain of the L^p generators)**.** *Assume a connected weighted graph allows for a compatible intrinsic metric. Then, for $p \in (1,\infty)$, the generator L_p is a restriction of Δ and*

$$D(L_p) = \{u \in L^p(X,m) \cap F \mid \Delta u \in L^p(X,m)\}.$$

We get furthermore a sufficient criterion for recurrence analogous to [Kar82, Theorem 3.5] and [Stu94, Theorem 3] which generalizes for example [DK87, Theorem 2.2], [RSV97, Corollary B], [Woe00, Lemma 3.12], [MUW12, Theorem 1.2] on graphs. For an abstract characterization of recurrence see [FŌT11, Ō92] (which is adapted to our setting in [Sch12]; confer [Soa94] for the normalized case). A proof of the corollary below is given in Section 3.4.

Corollary 1.4 (Recurrence)**.** *Assume a connected weighted graph allows for a compatible intrinsic metric. If*

$$\int_1^\infty \frac{r}{m(B_r)}dr = \infty,$$

then the graph is recurrent.

Remark 1.2. (a) The corollary above generalizes [HKMW, Theorem 1] to the case $p \in (1,\infty)$ and settles the question in [HKMW, Remark 3.6]. Moreover, it complements [KL12, Theorem 5].

(b) It would be interesting to know whether there is a Liouville type theorem for functions in $D(L_p)$ without the assumption of compatibility on the metric.

For vertices $x, y \in X$ that are connected by an edge, we denote a directed edge by xy and the positive symmetric edge weight by μ_{xy}. We define

$$\nabla_{xy} f = f(x) - f(y).$$

The following L^p Caccippoli-type inequality is a side product of our analysis. Such an equality was proven in [HS97, HJ13, RSV97] for bounded operators. The classical Caccioppoli inequality is the case $p = 2$, which can be found on graphs in [CG98, LX10, HKMW].

Theorem 1.5 (Caccioppoli-type inequality)**.** *Assume a connected weighted graph allows for a compatible intrinsic metric and $p \in (1,\infty)$. Then, there is $C > 0$ such*

that for every non-negative subharmonic function f and all $0 < r < R - 3s$

$$\sum_{x,y\in B_r} \mu_{xy}(f(x)\vee f(y))^{p-2}|\nabla_{xy}f|^2 \leq \frac{C}{(R-r)^2}\|f1_{B_R\setminus B_r}\|_p^p,$$

where s is the jump size of the intrinsic metric (see Section 2.3).

We prove the theorem in Section 3.3.

Remark 1.3. (a) The theorem above allows for a direct proof of Corollary 1.2, confer [HJ13, Corollary 3.1].

(b) For $p \geq 2$, we can strengthen the inequality by replacing $(f(x)\vee f(y))^{p-2}$ on the left hand side by $f^{p-2}(x) + f^{p-2}(y)$, see Remark 3.1.

Finally, we state three theorems where we put additional assumptions on the measure. They are also consequences of Theorem 1.1 and are proven in Section 3.5. We say a function $f : X \to \mathbb{R}$ *grows less than* a function $g : [0,\infty) \to (0,\infty)$ if there is $q \in (0,1)$ such that $\limsup_{r\to\infty} \|f1_{B_r}\|_\infty / g^q(r) < \infty$ and we say f *grows polynomially* if f grows less than a polynomial.

Theorem 1.6 (Finite measure)**.** *Assume a connected weighted graph allows for a compatible intrinsic metric and*

$$m(X) < \infty.$$

Then every non-negative subharmonic function f, that grows less than quadratic, is constant. In particular, $f \in L^\infty(X)$ implies that f is constant.

Theorem 1.7 (Finite moment measure)**.** *Assume a connected weighted graph allows for a compatible intrinsic metric ρ and the measure has a finite q-th moment, $q \in \mathbb{R}$, i.e.,*

$$\rho(\cdot, o) \in L^q(X\setminus\{o\}, m), \qquad \textit{for some } o \in X.$$

Then every non-negative subharmonic function f that grows less than $r \mapsto r^{q+2}$ is constant. In particular, $f \in L^\infty(X)$ implies f is constant if $q > -2$.

Theorem 1.8 (Decaying measure)**.** *Assume a connected weighted graph allows for a compatible intrinsic metric and there is $\beta > 0$ such that*

$$\limsup_{r\to\infty} \frac{1}{r^\beta} \log m(B_{r+1}\setminus B_r) < 0.$$

Then every non-negative subharmonic function f that grows polynomially is constant.

The paper is organized as follows. In the next section, we define the involved concepts and recall some basic inequalities. Section 3 is devoted to the proofs of the theorems and corollaries above. Finally, in Section 4, we give counter-examples to an L^1-Liouville type statement.

Throughout this paper C always denotes a constant that might change from line to line. Moreover, we use the convention that $\infty \cdot 0 = 0$, (which only appears in expressions such as $f^{-q}(x)\nabla_{xy}f$ with $f(x) = f(y) = 0$ and $q > 0$).

2. Set-up and preliminaries

2.1. **Weighted graphs.** Let X be a countable discrete set and $m : X \to (0, \infty)$. Then (X, m) becomes a measure space with a measure of full support. A graph over (X, m) is induced by an edge weight function $\mu : X \times X \to [0, \infty)$, $(x, y) \mapsto \mu_{xy}$ which is symmetric, has zero diagonal and satisfies

$$\sum_{y \in X} \mu_{xy} < \infty, \qquad x \in X.$$

If $\mu_{xy} > 0$ we write $x \sim y$ and let xy and yx be the oriented edges of the graph. Whenever we fix an orientation on the edges, we denote the set of oriented edges by E and denote the elements of E by e. We write $xy \subset A$ for a set $A \subseteq X$ if both of the vertices of the edge xy are contained in A, i.e., $x, y \in A$.

We refer to the triple (X, μ, m) as a *weighted graph*. We assume that the graph is *connected*, that is for every two vertices $x, y \in X$ there is a path $x = x_0 \sim x_1 \sim \ldots \sim x_n = y$.

The spaces $L^p(X, m)$, $p \in [1, \infty)$, and $L^\infty(X)$ are defined in the natural way. For $p \in [1, \infty)$, let p^* be its Hölder dual, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$.

2.2. **Laplacians and (sub)harmonic functions.** We define the *formal Laplacian* Δ on the *formal domain*

$$F = \{f : X \to \mathbb{R} \mid \sum_{y \in X} \mu_{xy} |f(y)| < \infty \text{ for all } x \in X\},$$

by

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \in X} \mu_{xy} (f(x) - f(y)).$$

Definition 2.1. A function $f : X \to \mathbb{R}$ is called *harmonic* (*subharmonic*, *superharmonic*) if $f \in F$ and $\Delta f = 0$, ($\Delta f \le 0$, $\Delta f \ge 0$).

We denote by L the positive selfadjoint restriction of Δ on $L^2(X, m)$ which arises from the closure Q of the quadratic form

$$u \mapsto \sum_{xy \in E} \mu_{xy} |\nabla_{xy} u|^2$$

on $C_c(X)$, the space of finitely supported functions, (for details see [KL12]). Since Q is Dirichlet form, the semigroup e^{-tL}, $t \ge 0$, extends to a C_0-semigroup on L^p, $p \in [1, \infty)$. We denote the generators of these semigroups by L_p.

2.3. **Intrinsic metrics.** Next, we introduce the concept of intrinsic metrics. A pseudo metric is a symmetric map $X \times X \to [0, \infty)$ with zero diagonal which satisfies the triangle inequality.

Definition 2.2 (Intrinsic metric)**.** A pseudo metric ρ on X is called an *intrinsic metric* if

$$\sum_{y \in X} \mu_{xy} \rho(x, y)^2 \le m(x), \qquad x \in X.$$

For a function $u : X \to \mathbb{R}$ such that the map $\Gamma(u) : x \mapsto \sum_{y\in X} \mu_{xy}|\nabla_{xy}u|^2$ takes finite values, $\Gamma(u)$ defines the energy measure of u. Thus, a pseudo metric ρ is intrinsic if the energy measures $\Gamma(\rho(x,\cdot))$, $x \in X$, are absolutely continuous with respect to m with Radon-Nikodym derivative $\frac{d}{dm}\Gamma(\rho(x,\cdot)) = \Gamma(\rho(x,\cdot))/m$ satisfying $\Gamma(\rho(x,\cdot))/m \leq 1$.

The concept of intrinsic metrics was developed since the natural graph distance is insufficient for the investigations of unbounded Laplacians, see [Woj09, Woj11, KLW]. This concept was already applied to various problems on graphs [BHK12, BKW, Fol11, Fol12, HKMW] and related settings [GHM12]. For a structural approach for regular Dirichlet forms see [FLW].

The *jumps size* s of a pseudo metric is given by

$$s := \sup\{\rho(x,y) \mid xy \in E\} \in [0,\infty].$$

From now on ρ always denotes an intrinsic metric and s denotes its jump size.

We fix a base point $o \in X$ which we suppress in notation and denote the distance balls by

$$B_r = \{x \in X \mid \rho(x,o) \leq r\}, \qquad r \geq 0.$$

Since ρ takes values in $[0,\infty)$, the results are indeed independent of the choice of o. For $U \subseteq X$, we write $B_r(U) = \{x \in X \mid \rho(x,y) \leq r \text{ for some } y \in U\}$, $r \geq 0$.

Define the weighted vertex degree $\mathrm{Deg} : X \to [0,\infty)$ by

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y\in X} \mu_{xy}, \qquad x \in X.$$

Definition 2.3 (Compatible metric)**.** A pseudo metric on X is called *compatible* if it has finite jump size and the restriction of Deg to every distance ball is bounded, i.e., $\mathrm{Deg}|_{B_r} \leq C(r) < \infty$ for all $r \geq 0$.

Example 2.4. (a) For any given weighted graph there is an intrinsic path metric defined by

$$\delta(x,y) = \inf_{x=x_0\sim\ldots\sim x_n=y} \sum_{i=0}^{n-1} (\mathrm{Deg}(x_i) \vee \mathrm{Deg}(x_{i+1}))^{-\frac{1}{2}}.$$

This intrinsic metric can be turned into an intrinsic metric δ_r with finite jump size $s = r$ by taking the path metric with edge weights $\delta \wedge r$. In general, neither δ_r nor δ is compatible.

(b) If the measure m is larger than the measure $n(x) = \sum_{y\in X} \mu_{xy}$, $x \in X$, then the natural graph distance (i.e., the path metric with edge weights 1) is an intrinsic metric which is compatible since $s = 1$ and $\mathrm{Deg} \leq 1$ in this case.

Remark 2.1. (a) In view of Example 2.4 (b) it is apparent that the setting of Theorem 1.1 is more general than [HJ13, Theorem 1.1].

(b) In [HKMW, Theorem A.1] a Hopf-Rinow type theorem is shown which states that for a locally finite graph a path metric is complete if and only if all balls are finite. Thus, compatibility can be seen as a completeness assumption of the graph.

(c) It is not hard to see that there are graphs that do not allow for a compatible intrinsic metric. However, to a given edge weight function μ and a pseudo metric ρ, we can always assign a minimal measure m such that ρ is intrinsic, i.e., let $m(x) = \sum_{y\in X} \mu_{xy}\rho(x,y)^2$, $x \in X$. If ρ already has finite jump size and all balls are finite, then ρ is automatically compatible.

In the subsequent, we will make use of the cut-off function $\eta = \eta_{r,R}$, $0 \leq r < R$, on X given by

$$\eta = 1 \wedge \Big(\frac{R - \rho(\cdot, o)}{R - r}\Big)_+.$$

Lemma 2.5. *Let $\eta = \eta_{r,R}$, $0 < r < R$, be given as above. Then*

(a) $\eta|_{B_r} \equiv 1$ *and* $\eta|_{X \setminus B_R} \equiv 0$.

(b) *For $x \in X$*

$$\sum_{y \in X} \mu_{xy} |\nabla_{xy} \eta|^2 \leq \frac{1}{(R-r)^2} 1_{B_{R+s} \setminus B_{r-s}}(x) m(x).$$

Proof. (a) is obvious from the definition of η and (b) follows directly from $|\nabla_{xy}\eta| \leq \frac{1}{R-r}\rho(x,y) 1_{B_{R+s} \setminus B_{r-s}}(x)$ for $x \sim y$ and the intrinsic metric property of ρ. □

2.4. **Green's formula, Leibniz rules and mean value inequalities.** We first prove a Green's formula which is slightly more general than the one in [HKMW].

Lemma 2.6 (Green's formula)**.** *Let $p \in [1, \infty)$, $U \subseteq X$ and assume* Deg *is bounded on U. Then for all f with $f 1_U \in L^p(X, m) \cap F$ and $g \in L^{p^*}(X, m)$ with $B_s(\operatorname{supp} g) \subseteq U$*

$$\sum_{x \in X} (\Delta f)(x) g(x) m(x) = \frac{1}{2} \sum_{x,y \in U} \mu_{xy} \nabla_{xy} f \nabla_{xy} g.$$

Proof. The formal calculation in the proof of Green's formula is a straightforward algebraic manipulation. To ensure that all involved terms converge absolutely one invokes Hölder's inequality and the boundedness assumption on Deg (confer the proof of Lemma 3.1 and 3.3 in [HKMW]). □

The following Leibniz rules follow from direct computation.

Lemma 2.7 (Leibniz rules)**.** *For all $x, y \in X$, $x \sim y$ and $f, g : X \to \mathbb{R}$*

$$\begin{aligned}\nabla_{xy}(fg) &= f(y)\nabla_{xy} g + g(x) \nabla_{xy} f \\ &= f(y)\nabla_{xy} g + g(y)\nabla_{xy} f + \nabla_{xy} f \nabla_{xy} g.\end{aligned}$$

A fundamental difference of Laplacians on graphs and on manifolds is the absence of a chain rule in the graph case. In particular, existence of a chain rule can be used as a characterization for a regular Dirichlet form to be strongly local. We circumvent this problem by using the mean value inequality from calculus instead. In particular, for a continuously differentiable function $\phi : \mathbb{R} \to \mathbb{R}$ and $f : X \to \mathbb{R}$, we have

$$\nabla_{xy}(\phi \circ f) = \phi'(\zeta) \nabla_{xy} f, \qquad \text{with } \zeta \in [f(x) \wedge f(y), f(x) \vee f(y)].$$

In this paper we will apply this to get estimates for the function $\phi : t \mapsto t^{p-1}$, $p \in (1, \infty)$. However, we need a refined inequality as it was already used in the proof of [HS97, Theorem 2.1]. For the convenience of the reader, we include a short proof here.

Lemma 2.8 (Mean value inequalities)**.** *For all $f : X \to \mathbb{R}$ and $x \sim y$ with $\nabla_{xy} f \geq 0$*

(a) $\nabla_{xy} f^{p-1} \geq \frac{1}{2}(f^{p-2}(x) + f^{p-2}(y)) \nabla_{xy} f$, *for $p \in [2, \infty)$,*

(b) $\nabla_{xy} f^{p-1} \geq C(f(x) \vee f(y))^{p-2} \nabla_{xy} f$, *for $p \in (1, \infty)$, where $C = (p-1) \wedge 1$.*

Proof. (a) Denote $a = f(y)$, $b = f(x)$ and as it is the only non-trivial case, we assume $0 < a < b$. Note that for $p \neq 1$

$$b^{p-1} - a^{p-1} = (b-a)(b^{p-2} + a^{p-2}) + ab(b^{p-3} - a^{p-3}).$$

Thus, the statement is immediate for $p \geq 3$ since the second term on the right side is positive in this case. Let $2 \leq p < 3$ and note $a^{p-3} > b^{p-3}$. The function $t \mapsto t^{2-p}$ is convex on $(0, \infty)$ and, thus, its image lies below the line segment connecting (b^{-1}, b^{p-2}) and (a^{-1}, a^{p-2}). Therefore,

$$\begin{aligned} a^{p-3} - b^{p-3} &\leq \frac{a^{p-3} - b^{p-3}}{(3-p)} = \int_{b^{-1}}^{a^{-1}} t^{2-p} dt \leq (a^{-1} - b^{-1}) \Big(\frac{(b^{p-2} - a^{p-2})}{2} + a^{p-2} \Big) \\ &= \frac{1}{2ab}(b-a)(a^{p-2} + b^{p-2}). \end{aligned}$$

From the equality in the beginning of the proof we now deduce the assertion in the case $2 \leq p < 3$.
(b) The case $p \geq 2$ follows from (a). The case $1 < p \leq 2$ in (b) follows directly from the mean value theorem. □

3. Proofs

In this section we prove the main theorems and the corresponding corollaries. It will be convenient to introduce the following orientation on the edges. Let a non-negative subharmonic function f be given. Let E be the antisymmetric set of oriented edges $e = e_+ e_-$ such that

$$\nabla_e f \geq 0, \quad \text{i.e., } \; f(e_+) \geq f(e_-).$$

3.1. The key estimate.

The lemma below is vital for the proof of Theorem 1.1 and Theorem 1.5.

Lemma 3.1. *Let $p \in (1, \infty)$, $0 \leq \varphi \in L^\infty(X)$ and $U = B_s(\operatorname{supp} \varphi)$. Assume* Deg *is bounded on U. Then for every non-negative subharmonic function f with $f 1_U \in L^p(X, m)$*

$$\sum_{e \in E, e \subset U} \mu_e f^{p-2}(e_+) \varphi^2(e_-) |\nabla_e f|^2 \leq C \sum_{e \in E, e \subset U} \mu_e f^{p-1}(e_+) \varphi(e_-) \nabla_e f |\nabla_e \varphi|,$$

where $C = 2/((p-1) \wedge 1)$.

Proof. From the assumptions $f 1_U \in L^p(X, m)$ and $\varphi \in L^\infty(X)$, we infer $\varphi f^{p-1} \in L^{p^*}(X, m)$ (as $p^* = p/(p-1)$). Thus, compatiblity of the metric implies applicability of Green's formula with f and $g = \varphi f^{p-1}$. We start by using non-negativity and subharmonicity of f before applying Green's formula (Lemma 2.6) and the first and second Leibniz rules (Lemma 2.7)

$$\begin{aligned} 0 &\geq \sum_{x \in X} (\Delta f)(x)(\varphi^2 f^{p-1})(x) m(x) = \sum_{e \in E, e \subset U} \mu_e \nabla_e f \nabla_e (\varphi^2 f^{p-1}) \\ &= \sum_{e \subset U} \mu_e \nabla_e f \big[\varphi^2(e_-) \nabla_e f^{p-1} + f^{p-1}(e_+) \nabla_e \varphi^2 \big] \\ &= \sum_{e \subset U} \mu_e \nabla_e f \big[\varphi^2(e_-) \nabla_e f^{p-1} + 2 f^{p-1}(e_+) \varphi(e_-) \nabla_e \varphi + f^{p-1}(e_+) |\nabla_e \varphi|^2 \big] \\ &\geq C \sum_{e \subset U} \mu_e f^{p-2}(e_+) \varphi^2(e_-) |\nabla_e f|^2 + 2 \sum_{e \subset U} \mu_e f^{p-1}(e_+) \varphi(e_-) \nabla_e f \nabla_e \varphi, \end{aligned}$$

where we dropped the third term in the third line since it is positive because of $\nabla_e f \geq 0$ and we estimated the first term on the right hand side using the mean value inequality, Lemma 2.8 (b). Hence, we obtain the statement of the lemma. (Note that absolute convergence of the two terms in the last line can be checked using Hölder's inequality and the assumptions $f1_U \in L^p(X,m)$, $\varphi \in L^\infty(X)$ and boundedness of Deg on U.) $\square$

3.2. **Proof of Karp's theorem.**

Proof of Theorem 1.1. Let $p \in (1,\infty)$. Assume $f1_{B_r} \in L^p(X,m)$ for all $r \geq 0$ since otherwise $\inf_{r_0} \int_{r_0}^\infty r/\|f1_{B_r}\|_p^p dr = 0$. Let $\eta = \eta_{r+s,R-s}$ with $0 < r < R - 3s$ (see Section 2.3). Then by Lemma 3.1 (applied with $\varphi = \eta$) we obtain (noting additionally that $\nabla_{xy}\eta = 0$, $x, y \in B_r$)

$$\sum_{e\subset B_R} \mu_e f^{p-2}(e_+)\eta^2(e_-)|\nabla_e f|^2 \leq C \sum_{e\subset B_R\setminus B_r} \mu_e f^{p-1}(e_+)\eta(e_-)\nabla_e f|\nabla_e \eta|.$$

Now, the Cauchy-Schwarz inequality and Lemma 2.5 yield

$$\begin{aligned}
&\Big(\sum_{e\subset B_R} \mu_e f^{p-2}(e_+)\eta^2(e_-)|\nabla_e f|^2\Big)^2 \\
&\qquad \leq C\Big(\sum_{e\subset B_R\setminus B_r} \mu_e f^p(e_+)|\nabla_e\eta|^2\Big)\Big(\sum_{e\subset B_R\setminus B_r} f^{p-2}(e_+)\eta^2(e_-)|\nabla_e f|^2\Big) \\
&\qquad \leq \frac{C}{(R-r)^2}\|f1_{B_R\setminus B_r}\|_p^p\left(\Big(\sum_{e\subset B_R} - \sum_{e\subset B_r}\Big) f^{p-2}(e_+)\eta^2(e_-)|\nabla_e f|^2\right),
\end{aligned}$$

where we also used $\sum_e \mu_e f^p(e_+)|\nabla_e \eta|^2 \leq \sum_{x,y} \mu_{xy} f^p(x)|\nabla_{xy}\eta|^2$. Let $R_0 \geq 3s$ be such that $f1_{B_{R_0}} \neq 0$ and denote

$$v(r) = \|f1_{B_r}\|_p^p, \qquad r \geq 0.$$

Moreover, for $j \geq 0$, let $R_j = 2^j R_0$ and

$$\begin{aligned}
\varphi_j &= \eta_{R_j+s,R_{j+1}-s}, \\
Q_{j+1} &= \sum_{e\subset B_{R_{j+1}}} \mu_e f^{p-2}(e_+)\varphi_j^2(e_-)|\nabla_e f|^2.
\end{aligned}$$

As $\varphi_{j-1} \leq \varphi_j$, the estimate above and $Q_j \leq Q_{j+1}$ imply

$$Q_j Q_{j+1} \leq Q_{j+1}^2 \leq C\frac{v(R_{j+1})}{(R_{j+1}-R_j)^2}(Q_{j+1} - Q_j), \quad j \geq 0.$$

Since $R_{j+1} = 2R_j$, dividing the above inequality by $\frac{v(R_{j+1})}{R_{j+1}} Q_j Q_{j+1}$ and adding C/Q_{j+1} yield

$$\frac{R_{j+1}^2}{v(R_{j+1})} + \frac{C}{Q_{j+1}} \leq \frac{C}{Q_j}$$

and, thus,

$$\frac{1}{C}\sum_{j=1}^\infty \frac{R_{j+1}^2}{v(R_{j+1})} \leq \frac{1}{Q_1}.$$

Now, the assumption $\int_{R_0}^{\infty} r/v(r)dr = \infty$ implies

$$\sum_{j=0}^{\infty} \frac{R_j^2}{v(R_j)} = \infty,$$

and, therefore, $Q_1 = 0$. As this is true for any R_0 large enough, we have

$$f^{p-2}(e_+)|\nabla_e f|^2 = 0,$$

for all edges e. For $p \geq 2$ connectedness clearly implies that f is constant. On the other hand, for $p \in (1, 2]$, we always have $f^{p-2}(e_+) > 0$ and, thus, f is constant. $\square$

3.3. **Proof of the Caccioppoli inequality.**

Proof of Theorem 1.5. Using Lemma 3.1 and the inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, $\varepsilon > 0$, we get

$$\begin{aligned}\sum_{e \in E} \mu_e f^{p-2}(e_+)\varphi^2(e_-)|\nabla_e f|^2 &\leq C \sum_{e \in E} \mu_e f^{p-1}(e_+)\varphi(e_-)\nabla_e f|\nabla_e \varphi| \\ &\leq \frac{1}{2} \sum_{e \in E} \mu_e f^{p-2}(e_+)\varphi^2(e_-)|\nabla_e f|^2 + C \sum_{e \in E} \mu_e f^p(e_+)|\nabla_e \varphi|^2.\end{aligned}$$

Letting $\varphi = \eta = \eta_{r+s,R-s}$ with $0 < r < R - 3s$ (from Section 2.3) and using Lemma 2.5, we arrive at

$$\sum_{e \in E} \mu_e f^{p-2}(e_+)|\nabla_e f|^2 \leq C \sum_{e \in E} \mu_e f^p(e_+)|\nabla_e \eta|^2 \leq \frac{C}{(R-r)^2} \|f 1_{B_R \setminus B_r}\|_p^p.$$

$\square$

Remark 3.1. In order to obtain the stronger statement for $p \in [2, \infty)$ mentioned in Remark 1.3 (b), we have to invoke Lemma 2.8 (a) in the proof of Lemma 3.1 instead of Lemma 2.8 (b) and proceed as in the proof.

3.4. **Proof of the Corollaries.** In this section we prove the corollaries.

Proof of Corollary 1.2 (Yau's L^p Liouville theorem). Clearly the integral in Theorem 1.1 diverges if $f \in L^p(X, m)$. $\square$

Proof of Corollary 1.3 (Domain of the L^p generators). Let $f \in L^p(X, m) \cap F$ be such that $(\Delta + 1)f = 0$. Since the positive and negative part f_+, f_- of f are non-negative, subharmonic and in $L^p(X, m)$, they must be constant by Corollary 1.2. By connectedness $f_\pm \equiv 0$ and, thus, $f \equiv 0$. Now, the proof of the corollary works literally line by line as the proof of [KL12, Theorem 5]. $\square$

Proof of Corollary 1.4 (Recurrence). Theorem 1.1 implies that any non-negative bounded subharmonic function f is constant provided $\inf_{r_0} \int_{r_0}^{\infty} r/m(B_r)dr = \infty$ (since $\|f 1_{B_r}\|_p^p \leq \|f\|_\infty m(B_r)$, $r \geq 0$). This implies that any bounded subharmonic function f is constant (since $g = f + \|f\|_\infty$ is a non-negative bounded subharmonic function which must be constant). This however implies that any bounded superharmonic function f is constant (since $-f$ is subharmonic). Finally, every non-negative superharmonic function f can be approximated by the bounded superharmonic functions $f_n = f \wedge n$. Now, according to [Sch12, Theorem 9] (confer [Soa94, Theorem 1.33] and [Stu94, Proof of Theorem 3]) recurrence is equivalent to all non-negative superharmonic functions of finite energy being constant. $\square$

3.5. **Applications to finite measure spaces.** The proofs of Theorem 1.6 and Theorem 1.7 are based on the following lemma.

Lemma 3.2. *Assume a connected weighted graph allows for a compatible intrinsic metric. If for a non-negative subharmonic function f and some $p \in (1, \infty)$*

$$f^p \rho^{-2}(\cdot, o) \in L^1(X \setminus \{o\}, m),$$

then f is constant.

Proof. For large r_0, we have

$$\int_{r_0}^{\infty} \frac{r}{\|f 1_{B_r}\|_p^p} dr \geq \int_{r_0}^{\infty} \frac{r}{r^2 \|f^p \rho^{-2}(\cdot, o) 1_{B_r \setminus \{o\}}\|_1 + C} dr \geq C \int_{r_0}^{\infty} \frac{1}{r} dr = \infty.$$

Hence, Theorem 1.1 implies the statement. □

Proof of Theorem 1.6 (Finite measure). If f grows less than quadratic, then there is $\varepsilon > 0$ such that $f^{1+\varepsilon}(x)\rho^{-2}(x, o) \leq C$ for $x \neq o$ and by $m(X) < \infty$ it follows $f^p \rho^{-2}(\cdot, o) \in L^1(X \setminus \{o\}, m)$ for $p = 1 + \varepsilon$. Hence, the theorem follows from the lemma above. □

Proof of Theorem 1.7 (Finite moment measure). If f grows less than $r \mapsto r^{q+2}$, then there is $\varepsilon > 0$ such that $f^{1+\varepsilon}\rho^{-2}(\cdot, o) \leq C\rho^q(\cdot, o)$ on $X \setminus \{o\}$. By the assumption $\rho(\cdot, o) \in L^q(X \setminus \{o\}, m)$ it follows $f^p \rho^{-2}(\cdot, o) \in L^1(X \setminus \{o\}, m)$ for $p = 1 + \varepsilon$. Hence, the assertion follows by the lemma above. □

Proof of Theorem 1.8 (Decaying measure). Since f grows less than polynomially there is $q > 0$ such that

$$\|f\|_p^p \leq C \sum_{x \in X} \rho^q(x, o) m(x) \leq C \sum_{r=0}^{\infty} r^q m(B_r \setminus B_{r-1}) < \infty$$

by the assumption on the measure. Hence, the theorem follows from Corollary 1.2. □

4. Counter-examples for $p = 1$

In this section, we deal with the borderline case of the L^p Liouville theorem, i.e., $p = 1$. We give two examples which show that there is no L^p Liouville theorem for positive subharmonic functions in the case $p \in (0, 1]$. That is, there is no L^1 Liouville theorem which is analogous to the situation in Riemannian geometry, where counter-examples were given by [Chu83, LS84]. Our first example is a graph of finite volume and the second is of infinite volume.

Example 4.1 (Finite volume)**.** Let $G = (X, E)$ be an infinite line, i.e., $X = \mathbb{Z}$ and $xy \in E$ iff $|x - y| = 1$ for $x, y \in \mathbb{Z}$. Define the edge weight by $\mu_{xy} = 2^{1-(|x| \vee |y|)}$ for $xy \in E$ and the measure m by $m(x) = (|x| + 1)^{-2} 2^{-|x|}$, $x \in \mathbb{Z}$ which implies $m(X) < \infty$. The intrinsic metric δ (introduced in Example 2.4) is compatible as it satisfies $\rho(x, x+1) \geq C(|x| + 1)^{-1}$ and, thus, $\sum_{x=-\infty}^{\infty} \rho(x, x+1) = \infty$. However, the function f defined as

$$f(x) = \operatorname{sign}(x)(2^{|x|} - 1), \quad x \in \mathbb{Z},$$

is harmonic and, clearly, $f \in L^p(X, m)$, $p \in (0, 1]$.

Example 4.2 (Infinite volume)**.** We can extend the example above to the infinite volume case. Let G be the graph from above and G' be a locally finite graph G' of infinite volume which allows for a compatible path metric. We glue G' to the vertex $x = 0$ of the graph G by identifying a vertex in G' with $x = 0$. Next, we extend the path metrics in a natural way and obtain (by renormalizing the edge weights at the edges around $x = 0$ if necessary) again a compatible intrinsic metric and the graph has infinite volume. Moreover, we extend f on G from above by zero to G' and obtain a harmonic function which is L^p, $p \in (0, 1]$.

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